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Ballisticity conditions for random walk in random environment

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Abstract

Consider a random walk in a uniformly elliptic i.i.d. random environment in dimensions $d \geq 2$. In 2002, Sznitman introduced for each $\gamma \in (0,1)$ the ballisticity conditions $(T)_\gamma$ and (T'), the latter being defined as the fulfilment of $(T)_\gamma$ for all $\gamma \in (0,1)$. He proved that (T') implies ballisticity and that for each $\gamma \in (0.5,1)$, $(T)_\gamma$ is equivalent to (T'). It is conjectured that this equivalence holds for all $\gamma \in (0,1)$. Here we prove that for $\gamma \in (\gamma_d, 1)$, where γ_d is a dimension dependent constant taking values in the interval $(0.366, 0.388)$, $(T)_\gamma$ is equivalent to (T'). This is achieved by a detour along the effective criterion, the fulfilment of which we establish by a combination of techniques developed by Sznitman giving a control on the occurrence of atypical quenched exit distributions through boxes.

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1 Introduction

We study the relationship between the ballisticity conditions (T') and $(T)_\gamma$, for $\gamma \in (0,1)$, introduced by Sznitman in [Szn02] for random walk in random environment (RWRE). Given a site $x \in \mathbb{Z}^d$, define the vector $\omega(x) := \{\omega(x,e) : e \in \mathbb{Z}^d, |e| = 1\}$ with $\omega(x,e) \in (0,1)$ and such that $\sum_{|e|=1} \omega(x,e) = 1$. We call the quantity $\omega := \{\omega(x) : x \in \mathbb{Z}^d\}$ an *environment*. Consider a Markov chain $\{X_n : n \geq 0\}$ on $\mathbb{Z}^d$ which jumps from each site $x \in \mathbb{Z}^d$ to the nearest neighbour site $x+e$ with probability $\omega(x,e)$. If the starting position of this chain is a site $x \in \mathbb{Z}^d$, denote by $P_{x,\omega}$ its law on $(\mathbb{Z}^d)^{\mathbb{N}}$. Assume that the environment ω is random and call μ its probability distribution. The *quenched law* of a RWRE is defined as the set of random probability measures $P_{x,\omega}$ with $x \in \mathbb{Z}^d$ under μ. The *averaged* or *annealed law* of a RWRE is the set of probability measures $P_x := \int P_{x,\omega}\, d\mu$ with $x \in \mathbb{Z}^d$. We will suppose that μ is a product measure, i.e. the random variables $\{\omega(x) : x \in \mathbb{Z}^d\}$ are i.i.d. with respect to μ. We will furthermore assume that μ is *uniformly elliptic* which means that there exists a constant $\kappa > 0$ such that

$$\mu(\inf_{|e|=1} \omega(0,e) > \kappa) = 1. \tag{1.1}$$

Given a vector $l \in \mathbb{S}^{d-1}$, a RWRE is called *transient in the direction* l if P_0-a.s.

$$\lim_{n\to\infty} X_n \cdot l = \infty.$$

Moreover, it is called *ballistic in the direction* l if P_0-a.s.

$$\liminf_{n\to\infty} \frac{X_n \cdot l}{n} > 0.$$

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Using renewal techniques it is not difficult to prove that ballisticity in the direction l is equivalent to the law of large numbers $\lim_{n\to\infty} \frac{X_n \cdot l}{n} = v$, with $v > 0$ deterministic. In dimension $d = 1$ it is well known that transience does not necessarily imply ballisticity. In dimensions $d \geq 2$ some fundamental questions about this model remain open.

Conjecture 1.1. *Transience in the direction l implies ballisticity in the direction l.*

Sznitman and Zerner [SZ99] and Zerner [Zer02] proved that the limit $\lim_{n\to\infty} X_n/n$ exists P_0-a.s. Subsequently, Sznitman [Szn02] introduced the conditions (T) and (T') related to a fixed direction $l \in \mathbb{S}^{d-1}$ which entail ballisticity. Let $\gamma \in (0,1)$. We say that condition $(T)_\gamma$ *relative to l is satisfied* (written as $(T)_\gamma|l)$ if for every l' in a neighborhood of l one has that

$$\limsup_{L\to\infty} L^{-\gamma} \log P_0(X_{T_{U_{l',b,L}}} \cdot l' < 0) < 0,$$

for all $b > 0$ and $U_{l',b,L} := \{x \in \mathbb{Z}^d : -bL < x \cdot l' < L\}$ with $T_{U_{l',b,L}}$ denoting the first exit time of $U_{l',b,L}$. We say that condition (T') *is satisfied relative to l* (written as $(T')|l$) if condition $(T)_\gamma|l$ holds for every $\gamma \in (0,1)$. We furthermore agree that condition (T) *relative to l is satisfied* and write $(T)|l$ if $(T)_\gamma|l$ holds for $\gamma = 1$. Let $\{e_j : 1 \leq j \leq d\}$ be the canonical generators of the additive group $\mathbb{Z}^d$. In dimension $d = 1$, $(T)|e_1$ is equivalent to transience in the direction e_1 (see Proposition 2.6 of [Szn01]) for which one has nice criteria at hand. Using an alternative characterisation of $(T)_\gamma$, in terms of transience in a given direction, one can in particular deduce that $(T)_\gamma|e_1$ is equivalent to transience in the direction e_1 for any $\gamma > 0$. In [Szn02], Sznitman proved that any RWRE in a uniformly elliptic environment which satisfies $(T')|l$, has a deterministic velocity

$$v := \lim_{n\to\infty} X_n/n, \quad P_0 - a.s.,$$

such that $v \cdot l > 0$, i.e. it is ballistic. He also showed that a central limit theorem is satisfied, so that

$$\frac{1}{\sqrt{n}}(X_{[n\cdot]} - [n\cdot]v)$$

converges under P_0 in law on $D(\mathbb{R}_+, \mathbb{R}^d)$ to a Brownian motion with non-degenerate covariance matrix. Furthermore, in [Szn02], the following conjecture for higher dimensions is stated:

Conjecture 1.2. *Let $d \geq 2$. For each $\gamma \in (0,1)$ and $l \in \mathbb{S}^{d-1}$, $(T)_\gamma|l$ is equivalent to $(T')|l$.*

Sznitman proved (see [Szn02]) that for each $\gamma \in (0.5, 1)$ and $l \in \mathbb{S}^{d-1}$, $(T)_\gamma|l$ is equivalent to $(T')|l$. The main result of this paper is the following.

Theorem 1.3. *Let $d \geq 2$ and*

$$\gamma_d := \frac{\sqrt{3d^2 - d} - d}{2d - 1}.$$

Then, for each $\gamma \in (\gamma_d, 1)$ and $l \in \mathbb{S}^{d-1}$, $(T)_\gamma|l$ is equivalent to $(T')|l$.

Remark 1.4. By direct inspection one observes that γ_d is monotonically decreasing in d. Therefore, $\gamma_\infty := \lim_{d\to\infty} \gamma_d = \frac{\sqrt{3}-1}{2}$ exists and we obtain

$$0.366 \approx \gamma_\infty < \gamma_d \leq \gamma_2 \approx 0.387.$$

The proof of Theorem 1.3 follows renormalization ideas introduced by Sznitman [Szn01], to control the probability of the existence of slowdown traps, and passes through the so called *effective criterion* introduced by him in [Szn02] as a tool which facilitates the checking of condition (T'). The effective criterion in a given direction is a sort of high dimensional version of the well known one-dimensional condition $\mathbb{E}(\rho) < 1$ [Sol75], which ensures ballisticity to the right of the RWRE and where $\rho = \omega(x, e_1)/\omega(x, -e_1)$. It introduces boxes $B := \{x \in \mathbb{Z}^d : x \in R((-(L-2), L+2) \times$

$(-\tilde{L},\tilde{L})^{d-1})\}$, where R is a rotation that fixes the origin and such that $R(e_1)=l$, with l the direction in the definition of condition (T'). $\tilde{L}$ will usually be chosen large and has to satisfy $3\sqrt{d}\le\tilde{L}<L^3$. It is important then to obtain a good control on the decay for large L of

$$\mathbb{P}(P_{0,\omega}(X_{T_B}\cdot l\ge L)\le e^{-L^\beta}), \tag{1.2}$$

where T_B is the first exit time from the box B and where $\beta\in(0,1)$ is an appropriately chosen parameter. Sznitman [Szn02] proves the equivalence for $\gamma\in(0.5,1)$, between $(T)_\gamma|l$ and $(T')|l$, establishing the equivalence between $(T)_\gamma|l$, the effective criterion in the direction l and $(T')|l$. To prove that $(T)_\gamma|l$ implies the effective criterion in the direction l, he bounds the quantity (1.2) for $\beta=\gamma$ through Chebychev's inequality. An ingredient in the proof of Theorem 1.3 of this paper, is the use of a renormalization step which starts from a *seed estimate*, both introduced by Sznitman in [Szn01], to obtain better controls of the quantity (1.2). Nevertheless, the materialisation into something useful via such an ingredient, requires a crucial step involving a careful decomposition of the quantity analogous to $\mathbb{E}(\rho)$ (in the one-dimensional case) entering the definition of the effective criterion.

In subsections 2.1 and 2.2, we recall this criterion and introduce some notation which will be needed afterwards, discussing the concept of asymptotic direction and stating Lemma 2.3, which provides a non-trivial control for the quantity (1.2). The proof of Theorem 1.3 is the subject of subsection 2.4. Section 3 is dedicated to proving Lemma 2.3. For this purpose, in subsection 3.1 we first recall a renormalization lemma of Sznitman [Szn01]. In subsection 3.2, we prove the seed estimate result, Lemma 3.3, which is a modification of Lemma 3.3 of [Szn01], under condition $(T)_\gamma$ instead of the stronger condition (T). Then, in subsection 3.3, these estimates are used to obtain a good control on (1.2) proving Lemma 2.3.

2 Preliminaries and proof of Theorem 1.3

Here we prove Theorem 1.3, showing that the effective criterion is satisfied with respect to the so called asymptotic direction. In subsection 2.1, we recall the definition of the effective criterion and its equivalence to the fulfilment of (T') as well as its equivalence to the fulfilment of $(T)_\gamma$ for some $\gamma\in(0.5,1)$ proved by Sznitman in [Szn02]. In subsection 2.2, we recall that $(T)_\gamma$ implies the a.s. existence of a deterministic asymptotic direction for the walk. Furthermore, we state Lemma 2.3, which gives a control on the quenched exit probabilities from boxes appearing in the definition of the effective criterion. The proof of this lemma is postponed to section 3. In subsection 2.3, departing from $(T)_\gamma$ for some $\gamma\in(\gamma_d,0.5]$, we prove the effective criterion with respect to the asymptotic direction. Finally, in subsection 2.4, we briefly explain how this implies Theorem 1.3.

2.1 Equivalence between (T') and the effective criterion

To prove Theorem 1.3 we will employ the so-called effective criterion. For positive numbers L, L' and $\tilde{L}$ as well as a space rotation R around the origin we use the box specification $\mathcal{B}(R,L,L',\tilde{L})$ to describe the set of boxes of the form $B:=\{x\in\mathbb{Z}^d: x\in R((-L,L')\times(-\tilde{L},\tilde{L})^{d-1})\}$. Furthermore, let

$$\rho_{\mathcal{B}}(\omega):=\frac{P_{0,\omega}(X_{T_B}\notin\partial_+B)}{P_{0,\omega}(X_{T_B}\in\partial_+B)}$$

where

$$\partial_+B:=\partial B\cap\{x\in\mathbb{Z}^d: l\cdot x\ge L', |R(e_i)\cdot x|\le\tilde{L}, i\in\{2,\dots,d\}\}.$$

Here, $\partial B:=\{x\in\mathbb{Z}^d: d(x,B)=1\}$ with $d(x,B)$ the distance from x to B in the 1-norm, and for $U\subset\mathbb{Z}^d$ we denote by T_U the first exit time $T_U:=\inf\{n\in\mathbb{N}: X_n\notin U\}$ with the convention $\inf\emptyset=\infty$. We will sometimes write ρ instead of $\rho_{\mathcal{B}}$ if the box we refer to is clear from the context. Note that due to the uniform ellipticity assumption, $\mathbb{P}$-a.s. we have $\rho\in(0,\infty)$. Given $l\in\mathbb{S}^{d-1}$, we say that the *effective criterion with respect to* l is satisfied if

$$\inf_{\mathcal{B},a}\{c_1(d)(\log\frac{1}{\kappa})^{3(d-1)}\tilde{L}^{d-1}L^{3(d-1)+1}\mathbb{E}\rho^a_{\mathcal{B}}\}<1.$$

Here, $c_1(d) > 1$ and $c_2(d) > 1$ are dimension dependent constants and a runs over $[0,1]$ while $\mathcal{B}$ runs over the box-specifications $\mathcal{B} = (R, L-2, L+2, \tilde{L})$ with R a rotation such that $R(e_1) = l$, $L \geq c_2(d)$, $3\sqrt{d} \leq \tilde{L} < L^3$.

The equivalence between $(T)_\gamma$ for any $\gamma \in (0.5,1)$ and (T') was established by Sznitman passing through the effective criterion.

Theorem 2.1 (Sznitman, [Szn02]). *For each $l \in \mathbb{S}^{d-1}$ the following conditions are equivalent.*

(a) There is a $\gamma \in (0.5,1)$ such that $(T)_\gamma|l$ is satisfied.

(b) The effective criterion with respect to l is satisfied.

(c) $(T')|l$ is satisfied.

To prove Theorem 1.3 we will also take advantage of the effective criterion with respect to a particular direction $\hat{v}$, called the *asymptotic direction*.

2.2 Asymptotic direction and atypical quenched exit distributions

Here we recall that under $(T)_\gamma|l$ the random walk has an asymptotic direction $\hat{v}$. As it will be explained, this implies that it will be enough to prove that $(T)_\gamma|l$ implies the effective criterion with respect to $\hat{v}$. In Corollary 1.5 of [Szn02], $(T)_\gamma|l$ is shown to be equivalent to the simultaneous fulfilment of the following conditions.

(i)

$$\{X_n : n \geq 0\} \text{ is transient in the direction } l.$$

(ii) For some $c > 0$,

$$E_0 \exp\{c \sup_{0 \leq n \leq \tau_1} |X_n|^\gamma\} < \infty. \tag{2.1}$$

Here, $|\cdot|$ denotes the Euclidean norm and τ_1 the regeneration time defined as the first time $X_n \cdot l$ obtains a new maximum and never goes below that maximum again, i.e.

$$\tau_1 := \inf\Big\{n \geq 1 : \sup_{0 \leq k \leq n-1} X_k \cdot l < X_n \cdot l \text{ and } \inf_{k \geq n} X_k \cdot l \geq X_n \cdot l\Big\}.$$

Transience in the direction l implies that τ_1 is P_0-a.s. finite, see [SZ99].

Due to (i), $(T)_\gamma|l$ implies that condition (a) of Theorem 1 in [Sim07] is fulfilled. Hence we have the existence of an asymptotic direction $\hat{v} \in \mathbb{S}^{d-1}$, i.e.

$$\lim_{n\to\infty} X_n/|X_n| = \hat{v} \quad P_0 - a.s. \tag{2.2}$$

Furthermore, as it is explained in Theorem 1.1 of [Szn02], under $(T)_\gamma|l$, $(T)_\gamma|l'$ holds if and only if $\hat{v} \cdot l' > 0$. Therefore, if we establish that $(T)_\gamma|l$ implies the effective criterion with respect to the asymptotic direction $\hat{v}$, by the equivalence between parts (b) and (c) of Theorem 2.1, we conclude that $(T')|\hat{v}$ holds, and hence $(T')|l$ also.

We now turn to two basic lemmas giving estimates on the occurrence of atypical quenched exit distributions for the RWRE. In the formulation of these results, B denotes the box $\{x \in \mathbb{Z}^d : x \in \hat{R}((-(L-2), L+2) \times (-3L, 3L)^{d-1})\}$ where $\hat{R}$ is a rotation mapping e_1 to $\hat{v}$, cf. (2.2). A *typical* quenched exit distribution for the RWRE gives a large probability to laws concentrated on the walk starting from 0 exiting the box B through the front part of the boundary $\partial_+ B$ defined in (2.1). The first lemma, whose proof we omit, is a direct consequence of Lemma 1.3 in [Szn02].

Lemma 2.2. *Let $l \in \mathbb{S}^{d-1}$ and assume that $(T)_\gamma|l$ is satisfied. Then*

$$-\delta_1 := \limsup_{L\to\infty} L^{-\gamma} \log P_0(X_{T_B} \notin \partial_+ B) < 0.$$

The second lemma will turn out to be a key estimate in the proof of Theorem 1.3. For this purpose we define the function

$$f(\beta) = d\Big(\beta - \frac{1}{1+\gamma}\Big)\frac{1+\gamma}{\gamma} \tag{2.3}$$

for $\beta \in ((1+\gamma)^{-1}, 1)$.

Lemma 2.3. *Assume that $(T)_\gamma|\hat{v}$ is satisfied. Then, if $\beta \in ((1+\gamma)^{-1}, 1)$, for any $c > 0$ and $\zeta \in (0, f(\beta))$ we have*

$$-\delta_2 := \limsup_{L\to\infty} L^{-\zeta} \log \mathbb{P}\big(P_{0,\omega}(X_{T_B} \in \partial_+ B) \le e^{-cL^\beta}\big) < 0.$$

Note that δ_2 may well depend on β, c and ζ. However, we usually do not name this dependence explicitly. The proof of this lemma involves the use of renormalization ideas beginning with a *seed estimate* lemma. We postpone it to section 3.

2.3 Proof of the effective criterion with respect to the asymptotic direction

As in the proof of Theorem 2.1 given by Sznitman in [Szn02], we will show that the quantity $\mathbb{E}\rho_{\mathcal{B}}^a$ decays as a stretched exponential as $L \to \infty$ for a suitable choice of a and $\mathcal{B}$. A key ingredient of the proof turns out to be the use of the renormalization ideas of Sznitman to obtain upper bounds for the probability of slowdown traps on the environment.

We will only consider the case $\gamma \le 0.5$. We set $a := L^{-\alpha}$ for some $\alpha \in (0,1)$ and consider the boxes $B := \{x \in \mathbb{Z}^d : x \in \hat{R}((-(L-2), L+2) \times (-3L, 3L)^{d-1})\}$ where $\hat{R}$ is a rotation mapping e_1 to $\hat{v}$, cf. (2.2). Uniform ellipticity yields $\mathbb{P}(P_{0,\omega}(X_{T_B} \in \partial_+ B) \le \kappa^{c(d)L}) = 0$ for some dimension dependent constant $c(d)$. Thus we can split $\mathbb{E}\rho_{\mathcal{B}}^a$ according to

$$\mathbb{E}\rho^a = (I) + (II) + (III), \tag{2.4}$$

where

$$\begin{aligned}(I) :=& \mathbb{E}\big(\rho^a, P_{0,\omega}(X_{T_B} \in \partial_+ B) > e^{-k_0 L^\gamma}\big),\\ (II) :=& \mathbb{E}\big(\rho^a, e^{-k_1 L^{\beta_1}} < P_{0,\omega}(X_{T_B} \in \partial_+ B) \le e^{-k_0 L^\gamma}\big)\end{aligned}$$

and

$$(III) := \sum_{j=1}^{n} \mathbb{E}\big(\rho^a, e^{-k_{j+1} L^{\beta_{j+1}}} < P_{0,\omega}(X_{T_B} \in \partial_+ B) \le e^{-k_j L^{\beta_j}}\big). \tag{2.5}$$

Here, n and k_0 as well as k_j and β_j for $j \in \{1, \dots, n+1\}$ are positive constants to be chosen later and satisfying

$$1 = \beta_{n+1} > \beta_n > \dots > \beta_1 > (1+\gamma)^{-1} \tag{2.6}$$

as well as k_{n+1} large enough. In fact, $k_1, \dots, k_n > 0$ can be chosen arbitrarily.

Lemma 2.4. *For all $L > 0$,*

$$(I) \le e^{k_0 L^{\gamma-\alpha} - \delta_1 L^{\gamma-\alpha} + o(L^{\gamma-\alpha})}.$$

Proof. By Jensen's inequality, we see that $(I) \le e^{ak_0 L^\gamma} P_0(X_{T_B} \notin \partial_+ B)^a$. The conclusion now follows from Lemma 2.2. $\square$

From this lemma it follows that if we choose

$$\alpha < \gamma \tag{2.7}$$

and

$$k_0 < \delta_1, \tag{2.8}$$

there exist positive constants c_1 and c_2 such that for all $L > 0$,

$$(I) \le c_1 e^{-c_2 L^{\gamma-\alpha}}.$$

Lemma 2.5. *For all $L > 0$,*

$$(II) \leq e^{k_1 L^{\beta_1 - \alpha} - \delta_1 L^\gamma + o(L^\gamma)}.$$

Proof. Note that

$$\begin{aligned}(II) &\leq e^{ak_1 L^{\beta_1}} \mathbb{E}(P_{0,\omega}(X_{T_B} \notin \partial_+ B)^a, P_{0,\omega}(X_{T_B} \notin \partial_+ B) \geq 1 - e^{-k_0 L^\gamma}) \\ &\leq e^{k_1 L^{\beta_1 - \alpha}} P_0(X_{T_B} \notin \partial_+ B)(1 - e^{-k_0 L^\gamma})^{-1},\end{aligned}$$

where to obtain the second line we used Chebychev's inequality. ☐

From this lemma we see that choosing

$$\beta_1 < 2\gamma \tag{2.9}$$

and α satisfying

$$\alpha \in (\beta_1 - \gamma, \gamma), \tag{2.10}$$

one then has that there exist positive constants c_1 and c_2 such that for all $L > 0$,

$$(II) \leq c_1 e^{-c_2 L^\gamma}.$$

Now, to control the third term, we use the following lemma.

Lemma 2.6. *Let the β_j's be chosen as in (2.6). Then, for all $L > 0$, $j \in \{1, \dots, n\}$ and $\zeta \in (0, f(\beta_j))$,*

$$\mathbb{E}\big(\rho^a, e^{-k_{j+1} L^{\beta_{j+1}}} \leq P_{0,\omega}(X_{T_B} \in \partial_+ B) \leq e^{-k_j L^{\beta_j}}\big) \leq e^{k_{j+1} L^{\beta_{j+1} - \alpha} - \delta_2 L^\zeta + o(L^\zeta)}$$

where f is defined as in (2.3).

Proof. We estimate

$$\begin{aligned}\mathbb{E}(\rho^a, &e^{-k_{j+1} L^{\beta_{j+1}}} \leq P_{0,\omega}(X_{T_B} \in \partial_+ B) \leq e^{-k_j L^{\beta_j}}) \\ &\leq e^{k_{j+1} L^{\beta_{j+1} - \alpha}} \mathbb{P}(P_{0,\omega}(X_{T_B} \in \partial_+ B) \leq e^{-k_j L^{\beta_j}}).\end{aligned}$$

Since $\beta_j > (1+\gamma)^{-1}$, the application of Lemma 2.3 yields the result. ☐

To prove the effective criterion with respect to $\hat{v}$, it is enough to prove that the terms (I), (II) and (III) of the decomposition (2.4) decay stretched exponentially. As follows from the discussions subsequent to Lemmas 2.4 and 2.5, for (I) and (II) this is achieved by respecting (2.7), (2.8), (2.9) and (2.10). It therefore remains to deal with (III). Since we may choose $\alpha < \gamma$ arbitrarily close to γ, Lemma 2.6 assures the desired decay once the following set of inequalities is fulfilled:

$$\left.\begin{array}{rl} \frac{1}{1+\gamma} < \beta_1 & < 2\gamma, \\ \frac{1}{1+\gamma} < \beta_2 & < \gamma + f(\beta_1), \\ \frac{1}{1+\gamma} < \beta_3 & < \gamma + f(\beta_2), \\ & \vdots \\ \frac{1}{1+\gamma} < \beta_n & < \gamma + f(\beta_{n-1}), \\ 1 & < \gamma + f(\beta_n). \end{array}\right\} \tag{2.11}$$

Now define $F(x) := \gamma + f(x)$ and for $k \geq 1$, $F^{(k)}(x) := F \circ F^{(k-1)}(x)$ with $F^{(0)}(x) = x$. Then in particular (2.11) is fulfilled if

$$\frac{1}{1+\gamma} < \beta_j < F^{(j-1)}(\beta_{j-1}), \quad j \in \{1, \dots, n+1\}, \tag{2.12}$$

with $\beta_0 := 2\gamma$. Therefore, it is enough to choose γ such that $(1+\gamma)^{-1} < 2\gamma$ and $(F^{(j)}(2\gamma))_{j\geq 0}$ forms an increasing sequence with

$$1 < \lim_{j\to\infty} F^{(j)}(2\gamma). \tag{2.13}$$

In this case we can choose the constants appearing in (2.5) according to $n := \inf\{j \in \mathbb{N} : F^{(j)}(\gamma) > 1\}$ and β_j as large as permitted by (2.12).

Now in order to check (2.13) we solve the equation $x = F(x)$ for x to obtain the (unstable) fixed-point

$$x^* := \frac{d-\gamma^2}{(1+\gamma)d-\gamma}.$$

Thus, we observe that it is sufficient to have $2\gamma > x^* > (1+\gamma)^{-1}$ in order for (2.13) to be fulfilled. For $\gamma \in (0, 0.5]$, it is easy to check that the second inequality is satisfied. Furthermore, the first inequality

$$2\gamma > \frac{d-\gamma^2}{(1+\gamma)d-\gamma},$$

is clearly true whenever

$$\gamma > \gamma_d = \frac{-2d+\sqrt{12d^2-4d}}{2(2d-1)}.$$

2.4 Proof of Theorem 1.3

By Theorem 2.1 of Sznitman [Szn02], it is enough to consider the case in which $\gamma \in (\gamma_d, 0.5]$. Assume that $(T)_\gamma|l$ holds for some $l \in \mathbb{S}^{d-1}$. It follows from Theorem 1.1 of [Szn02] that $l\cdot\hat{v} > 0$ and that $(T)_\gamma|l'$ is satisfied if and only if $\hat{v}\cdot l' > 0$. In particular, $(T)_\gamma|\hat{v}$ holds. In the previous subsection we proved that if $(T)_\gamma|\hat{v}$ is satisfied for some $\gamma \in (\gamma_d, 0.5]$, then the effective criterion is satisfied with respect to the asymptotic direction $\hat{v}$. Now, by the equivalence between parts (b) and (c) of Theorem 2.1, it follows that $(T')|\hat{v}$ is satisfied. Since $\hat{v}\cdot l > 0$, it follows that $(T')|l$ is satisfied.

3 Atypical quenched exit distribution estimates

The aim of this section is to prove Lemma 2.3. We will apply Lemma 3.2 of [Szn01], which we recall in subsection 3.1 and a modification of Lemma 3.3 of the same paper, which we prove in subsection 3.2. In subsection 3.3 we show how these results imply Lemma 2.3.

3.1 Sznitman's renormalization lemma

We introduce for $\beta, L > 0$ and $w \in \mathbb{Z}^d$ the notation

$$X_{\beta,L}(w) := -\log \inf_{x\in B_{1,\beta,L}(w)} P_{x,\omega}(X_{T_{B_{2,\beta,l(w)}}} \in \partial^* B_{2,\beta,L}(w)),$$

where

$$B_{1,\beta,L}(w) := \big\{x \in \mathbb{Z}^d : x \in \hat{R}(w + [0,L] \times [0, L^\beta]^{d-1})\big\},$$
$$B_{2,\beta,L}(w) := \big\{x \in \mathbb{Z}^d : x \in \hat{R}(w + (-dL^\beta, L] \times (-dL^\beta, (d+1)L^\beta)^{d-1})\big\}$$

and

$$\partial^* B_{2,\beta,L}(w) := \partial B_{2,\beta,L}(w) \cap B_{1,\beta,L}(w + Le_1).$$

We now recall the statement of the renormalization result of [Szn01].

Lemma 3.1 (Sznitman, [Szn01]). *Assume $d \geq 2$ and (1.1). Assume that $\beta_0 \in (0,1)$ and f_0 is a positive function defined on $[\beta_0, 1)$ such that*

$$f_0(\beta) \geq f_0(\beta_0) + \beta - \beta_0 \quad \textit{for } \beta \in [\beta_0, 1)$$

and, for $\beta \in [\beta_0, 1), \zeta < f_0(\beta)$,

$$\lim_{\beta' \uparrow \beta} \limsup_{L \to \infty} L^{-\zeta} \log \mathbb{P}(X_{\beta_0, L}(0) \geq L^{\beta'}) < 0.$$

Denote by f *the linear interpolation on* $[\beta_0, 1]$ *of the value* $f_0(\beta_0)$ *at* β_0 *and the value* d *at* 1. *Then, for* $\beta \in [\beta_0, 1)$ *and* $\zeta < f(\beta)$,

$$\lim_{\beta' \uparrow \beta} \limsup_{L \to \infty} L^{-\zeta} \log P(X_{\beta, L}(0) \geq L^{\beta'}) < 0. \tag{3.1}$$

3.2 Seed estimate under condition $(T)_\gamma$

To prove Lemma 2.3 we will apply Lemma 3.1. But we need to find an optimal function f_0 for which the assumption of this lemma are satisfied. That is the content of the so called seed estimate, Lemma 3.3, which we will prove in this subsection. This result is analogous to Lemma 3.3 of [Szn01], which assumes (T') and in turn relies on Lemma 2.3 of the same paper which gives a control for the annealed probability for the fluctuations of the projection on the orthogonal complement of $\hat{v}$ of the walk. Since we will instead only assume condition $(T)_\gamma$, we need some control analogous to Lemma 2.3 of [Szn01]. For completeness, we state such result, which was also proved by Sznitman, as Theorem A.2 in [Szn02]. First we introduce as in [Szn02], for $z \in \mathbb{Z}^d$ the following notation for the orthogonal projection on the orthogonal subspace of $\hat{v}$

$$\pi(z) := z - z \cdot \hat{v}\hat{v},$$

and for $u \in \mathbb{R}$ and $l \in \mathbb{S}^{d-1}$, the last visit of X_n to $\{x \in \mathbb{Z}^d : l \cdot x \leq u\}$ is denoted by

$$L_u^l := \sup\{n \geq 0 : X_n \cdot l \leq u\}.$$

Theorem 3.2 (Sznitman, [Szn02])**.** *Assume that for some* $\gamma \in (0, 1]$, $(T)_\gamma$ *holds with respect to* $l \in \mathbb{S}^{d-1}$. *Then for any* $c > 0$, $\rho \in (0.5, 1]$,

$$\limsup_{u \to \infty} u^{-(2\rho - 1) \wedge \gamma\rho} \log P_0\Big(\sup_{0 \leq n \leq L_u^l} |\pi(X_n)| \geq cu^\rho \Big) < 0.$$

Now, with the help of Theorem 3.2, we will prove the following lemma, following closely the proof of Lemma 3.3 of [Szn01]. We also include the whole proof in the paper for completeness.

Lemma 3.3. *Let* $\gamma \in (0, 1)$ *and assume that condition* $(T)_\gamma|\hat{v}$ *is satisfied. Then, for each* $\beta_0 \in (1/2, 1)$, *we have that for every* $\rho > 0$ *and* $\beta \in [\beta_0, 1)$

$$\limsup_{L \to \infty} L^{-(\beta + \beta_0 - 1) \wedge \gamma\beta_0} \log \mathbb{P}(X_{\beta_0, L} \geq \rho L^\beta) < 0. \tag{3.2}$$

Proof. We proceed as in the proof of Lemma 3.3 of [Szn01], taking advantage of Theorem 3.2. Define $\chi := \beta_0 + 1 - \beta \in (\beta_0, 1]$,

$$L_0 := \frac{L - \eta L^{\beta_0}}{\lfloor L^{1-\chi} \rfloor}$$

as well as

$$\tilde{B}_1(w) := \big\{x \in \mathbb{Z}^d : x \in \hat{R}([0, L_0] \times [0, L^{\beta_0}]^{d-1})\big\}$$

and

$$\tilde{B}_2(w) := \big\{x \in \mathbb{Z}^d : x \in \hat{R}((-dL^{\beta_0}, L_0] \times (-\eta L^{\beta_0}, (1+\eta)L^{\beta_0})^{d-1})\big\}$$

for $w \in \mathbb{Z}^d$ and $\eta > 0$. Keeping to the notation of [Szn01] we say that a point $w \in \mathbb{Z}^d$ is *bad* if

$$\inf_{x \in \tilde{B}_1(w)} P_{x,\omega}(X_{T_{\tilde{B}_2(w)}} \in \partial_+ \tilde{B}_2(w)) < 1/2$$

and *good* otherwise. Here $\partial_+ \tilde{B}_2$ is defined as in (2.1). By Chebyshev's inequality

$$\mathbb{P}(w \text{ is bad}) \le 2^{d+1} L_0 L^{(d-1)\beta_0} \Big(P_0 \big(\sup_{0 \le n \le T^{\hat{v}}_{L_0}} |\pi(X_n)| \ge \eta L^{\beta_0} \big) + P_0 (T^{-\hat{v}}_{dL^{\beta_0}} < \infty) \Big), \tag{3.3}$$

where for $v \in \mathbb{R}^d$ and $L \in \mathbb{R}$ we employed the stopping time $T^v_L := \inf\{n \in \mathbb{N} : X_n \cdot v \ge L\}$. The first summand can now be estimated via Theorem 3.2. This yields

$$0 > \limsup_{L \to \infty} L^{-(2\beta_0 - \chi) \wedge \gamma\beta_0} \log P_0 \big(\sup_{0 \le n \le T^{\hat{v}}_{L_0}} |\pi(X_n)| \ge \eta L^{\beta_0} \big). \tag{3.4}$$

The second summand is estimated as in [Szn01] yielding due to (2.1) that

$$\limsup_{L \to \infty} L^{-\gamma\beta_0} \log P_0 (T^{-\hat{v}}_{dL^{\beta_0}} < \infty) < 0 \tag{3.5}$$

Inserting the definition of χ, (3.4), (3.5) and (3.3) gives the estimate

$$\limsup_{L \to \infty} L^{-(\beta + \beta_0 - 1) \wedge \gamma\beta_0} \log P_0(w \text{ is bad}) < 0. \tag{3.6}$$

We now consider a certain set of trajectories starting in $B_{1,\beta,L}(0)$ and leaving $B_{2,\beta,L}(0)$ via $\partial^* B_{2,\beta,L}(0)$. We then show that if the points jL_0e_1, $j \in \{0, \ldots, \lfloor L^{1-\chi} \rfloor\}$ are all good, the above set of trajectories has a probability larger than $e^{-\rho L^\beta}$ to occur, hence it remains only to estimate the probability that one of the jL_0e_1 is bad in an adequate way.

Now to describe the above mentioned set of trajectories consider a walk starting in $B_{1,\beta_0,L}(0) \cap \tilde{B}_1(j_0 L_0 e_1)$, some $j_0 \in \{0, \ldots, \lfloor L^{1-\chi} \rfloor - 1\}$ and let it leave $\tilde{B}_2(j_0 L_0 e_1)$ through $\partial_+ \tilde{B}_2(j_0 L_0 e_1)$. From this point of exit, the walk can reach $\tilde{B}_1((j_0+1)L_0e_1)$ within $c(d)\eta L^{\beta_0}$ steps and stay within $\tilde{B}_2((j_0+1)L_0e_1)$ along this way. We then assume the walk to exit $\tilde{B}_2((j_0+1)L_0e_1)$ in the same way as $\tilde{B}_2(j_0L_0e_1)$, return to the box $\tilde{B}_1((j_0+2)L_0e_1)$ in the same way as before to $\tilde{B}_1((j_0+1)L_0e_1)$ and so on. When reaching $\partial_+ \tilde{B}_2((\lfloor L^{1-\chi} \rfloor - 1)L_0e_1)$, we want the walk to enter $\tilde{B}_1(\lfloor L^{1-\chi} \rfloor L_0 e_1)$ without leaving $B_{2,\beta_0,L}(0) \cap \tilde{B}_2(\lfloor L^{1-\chi} \rfloor L, e_1)$ and then exit $B_{2,\beta_0,L}(0)$ through $\partial^* B_{2,\beta_0,L}(0)$. These two requirements can be met within $2c(d)\kappa^{L^{\beta_0}}$ steps.

Now assume that the points jL_0e_1, $j \in \{0, \ldots, \lfloor L^{1-\chi} \rfloor - 1\}$ are all good. The strong Markov property applied to each of the exit and entrance times of the trajectories described above, yields

$$\begin{aligned} P_{x,\omega}(X_{T_{B_{2,\beta_0,L}(0)}} \in \partial_+ B_{2,\beta_0,L}(0)) &\ge \Big(\frac{1}{2} \kappa^{c(d)\eta L^{\beta_0}} \Big)^{L^{1-\chi}} \kappa^{2c(d)\eta L^{\beta_0}} \\ &> \exp\{-\rho L^\beta\} \end{aligned}$$

for $\eta > 0$ small enough and all $x \in B_{1,\beta_0,L}(0)$. Thus, translation invariance of the environment yields $\mathbb{P}(X_{\beta_0,L} \ge \rho L^\beta) \le L^{1-\chi} \mathbb{P}(0 \text{ is bad})$, which in combination with (3.6) finishes the proof. $\square$

3.3 Proof of Lemma 2.3

For $\varepsilon > 0$ small enough we define $f_{0,\varepsilon} : [(1+\gamma)^{-1} + \varepsilon, 1) \to [0,1]$ by

$$f_{0,\varepsilon}(\beta) := \beta - (1+\gamma)^{-1}.$$

Then, by Lemma 3.2 the assumptions of Lemma 3.1 are satisfied for $\beta_0 := (1+\gamma)^{-1} + \varepsilon$ and $f_0 := f_{0,\varepsilon}$. Therefore, defining $f_\varepsilon : [(1+\gamma)^{-1} + \varepsilon, 1) \to [0,1]$ as the linear interpolation between the value ε at $(1+\gamma)^{-1} + \varepsilon$ and the value d at 1, i.e.

$$f_\varepsilon(\beta) := d \Big(\beta - \frac{1}{1+\gamma} - \varepsilon \Big) \frac{(1+\gamma)\left(1 - \frac{\varepsilon}{d}\right)}{\gamma - \varepsilon - \gamma\varepsilon} + \varepsilon,$$

by Lemma 3.1 we can see that for $\beta \in [(1+\gamma)^{-1} + \varepsilon, 1)$ and $\zeta < f_\varepsilon(\beta)$, (3.1) holds.

Using the strong Markov property applied at $T_{B_{2,\beta,L}(0)}$ we obtain for all $\beta \in (0,1)$ and $L > 0$ large enough

$$P_{0,\omega}(X_{T_{B_{2,\beta,L}(0)}} \in \partial^* B_{2,\beta,L}(0)) \le P_{0,\omega}(X_{T_B} \in \partial_+ B) \cdot \kappa^{-2c(d)}$$

and hence in combination with the previously established version of (3.1), for any $\beta \in [(1+\gamma)^{-1}+\varepsilon, 1)$ and $\zeta < f_\varepsilon(\beta)$,

$$\begin{aligned}
&\limsup_{L\to\infty} L^{-\zeta} \log \mathbb{P}(P_{0,\omega}(X_{T_B} \in \partial_+ B) \le e^{-cL^\beta}) \\
&\le \limsup_{L\to\infty} L^{-\zeta} \log \mathbb{P}\left(X_{\beta,L}(0) \ge cL^\beta + 2c(d)\log\kappa\right) \\
&= -\delta_2 < 0.
\end{aligned}$$

Letting $\varepsilon \downarrow 0$, this proves Lemma 2.3.

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References

[Sim07] François Simenhaus. Asymptotic direction for random walks in random environments. *Ann. Inst. H. Poincaré*, 43(6):751–761, 2007.

[Sol75] Fred Solomon. Random walk in a random environment. *Ann. Probab.*, 3:1–31, 1975.

[SZ99] Alain-Sol Sznitman and Martin Zerner. A law of large numbers for random walks in random environment. *Ann. Probab.*, 27(4):1851–1869, 1999.

[Szn01] Alain-Sol Sznitman. On a class of transient random walks in random environment. *Ann. Probab.*, 29(2):724–765, 2001.

[Szn02] Alain-Sol Sznitman. An effective criterion for ballistic behavior of random walks in random environment. *Probab. Theory Related Fields*, 122(4):509–544, 2002.

[Zer02] Martin P. W. Zerner. A non-ballistic law of large numbers for random walks in i.i.d. random environment. *Electron. Comm. Probab.*, 7:191–197 (electronic), 2002.